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LETTER TO THE EDITOR

Symmetries for certain coupled nonlinear Schrödinger equations

C Sophocleous

Department of Mathematics and Statistics, University of Cyprus, Nicosia, PO Box 537, Cyprus

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Abstract. In this letter generalized symmetries are sought for certain coupled nonlinear Schrödinger equations. It is shown that all Lie–Bäcklund symmetries but one are equivalent to Lie point symmetries.

1. Introduction

We consider the coupled nonlinear Schrödinger equations (CNLS)

$$\begin{aligned} F_1 \; &: \quad \mathrm{i}u_t + u_{xx} + u^2\bar{u} + huv\bar{v} + \alpha u + \beta v = 0 \\ F_2 \; &: \quad \mathrm{i}v_t + v_{xx} + v^2\bar{u} + hvu\bar{u} - \alpha v + \beta u = 0 \end{aligned} \tag{1}$$

where $\bar{u}$ and $\bar{v}$ denote the conjugates of the dependent variables u and v, respectively, and the subscripts denote derivatives with respect to the indicated variables. Many descriptions of two nonlinearly coupled modulated wavetrains, particularly in fibre optics lead to such a system of nonlinear partial differential equations. The dependent variables $u(x,t)$ and $v(x,t)$ are two complex mode amplitudes in a birefringent fibre, while h, α and β are real constants. It is found [1] that equations (1) are completely integrable in the case where $\alpha = \beta = 0$ and $h = 1$. For this Manakov [2] derived explicit soliton solutions.

The technology of optical fibres for long distance communication and signal processing has developed rapidly. A large number of CNLS systems have arisen and been examined analytically and numerically. The system (1) has been investigated numerically by Trillo *et al* [3]. In this paper we search for generalized (Lie–Bäcklund) transformations of the form

$$\begin{aligned} &x' = x \qquad t' = t \\ &u' = u + \epsilon\eta(x, t, u, u_1, u_2, \ldots\bar{u}, \bar{u}_1, \ldots v, v_1, \ldots\bar{v}, \bar{v}_1, \ldots) + \mathrm{o}(\epsilon^2) \\ &v' = v + \epsilon\xi(x, t, u, u_1, u_2, \ldots\bar{u}, \bar{u}_1, \ldots v, v_1, \ldots\bar{v}, \bar{v}_1, \ldots) + \mathrm{o}(\epsilon^2) \end{aligned} \tag{2}$$

where $u_i = \partial^i u/\partial x^i$ $(i = 1, 2, 3, 4, \ldots)$, and similarly for $\bar{u}_i$, v_i and $\bar{v}_i$, which leave the system (1) invariant. Clearly, from transformations (2) we have $\bar{u}' = \bar{u} + \epsilon\bar{\eta} + \mathrm{o}(\epsilon^2)$ and $\bar{v}' = \bar{v} + \epsilon\bar{\xi} + \mathrm{o}(\epsilon^2)$. Such transformations include all Lie groups of point transformations of the form

$$\begin{aligned} x' &= x + \epsilon X(x, t) + \mathrm{o}(\epsilon^2) \\ t' &= t + \epsilon T(t) + \mathrm{o}(\epsilon^2) \\ u' &= u + \epsilon U(x, t, u, \bar{u}, v, \bar{v}) + \mathrm{o}(\epsilon^2) \\ v' &= v + \epsilon V(x, t, u, \bar{u}, v, \bar{v}) + \mathrm{o}(\epsilon^2)\,. \end{aligned} \tag{3}$$

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In fact, in many cases the transformations of the form (2) obtained are equivalent to transformations of the form (3). A generalized symmetry which is not equivalent to point transformation is called *proper Lie–Bäcklund symmetry.* The equivalency decomposition [4] which connect the two forms of transformations is given by

$$\begin{aligned} \eta &= U - u_x X - u_t T \\ \xi &= V - v_x X - v_t T\,. \end{aligned} \tag{4}$$

The constants α and β in the system (1) can both be taken to be zero or non-zero or either of the two to be zero. In all cases the resulting CNLS equations have a number of applications in the study of optical fibres. We shall refer to these CNLS equations as CNLS(A) when $\alpha \neq 0$ and $\beta \neq 0$, CNLS(B) when $\alpha \neq 0$ and $\beta = 0$, CNLS(C) when $\alpha = 0$ and $\beta \neq 0$ and CNLS(D) when $\alpha = \beta = 0$.

In the following analysis we omit all heavy calculations which have been greatly facilitated by the computer algebraic package REDUCE [5].

2. Symmetries

A symmetry of CNLS (1) of the form (2) satisfies

$$\Gamma^{(2)} F_1 = 0 \qquad \Gamma^{(2)} F_2 = 0 \tag{5}$$

where $\Gamma^{(2)}$ is the second extended infinitesimal generator and is given by the formula

$$\begin{aligned} \Gamma^{(2)} = \eta\frac{\partial}{\partial u} + \xi\frac{\partial}{\partial v} + \mathrm{D}_x\eta\frac{\partial}{\partial u_x} + \mathrm{D}_t\eta\frac{\partial}{\partial u_t} + \mathrm{D}_x\xi\frac{\partial}{\partial v_x} + \mathrm{D}_t\xi\frac{\partial}{\partial v_t} \\ +(\mathrm{D}_x)^2\eta\frac{\partial}{\partial u_{xx}} + \mathrm{D}_x\mathrm{D}_t\eta\frac{\partial}{\partial u_{xt}} + (\mathrm{D}_t)^2\eta\frac{\partial}{\partial u_{tt}} \\ +(\mathrm{D}_x)^2\xi\frac{\partial}{\partial v_{xx}} + \mathrm{D}_x\mathrm{D}_t\xi\frac{\partial}{\partial v_{xt}} + (\mathrm{D}_t)^2\xi\frac{\partial}{\partial v_{tt}} \end{aligned}$$

where D_x and D_t are the total derivatives with respect to x and t, respectively. The system (5) are two identities which involve the variables $x, t, u, v, \bar{u}, \bar{v}$ and their derivatives. Equations (1) can be solved for u_t and v_t, respectively, to give

$$\begin{aligned} u_t &= \mathrm{i}(u_{xx} + u^2\bar{u} + huv\bar{v} + \alpha u + \beta v) \\ v_t &= \mathrm{i}(v_{xx} + v^2\bar{v} + hvu\bar{u} - \alpha v + \beta u)\,. \end{aligned} \tag{6}$$

Employment of (6) enables us to evaluate all derivatives with respect to t in terms of $u, v, \bar{u}, \bar{v}$ and their derivatives with respect to x. That is, if we continually differentiate (6) we obtain a number of differential consequences which are used to eliminate u_{tx}, u_{tt}, u_{txx}, u_{ttx}, etc. Hence, identities (5) take the form

$$\begin{aligned} E_1(x, t, u, v, \bar{u}, \bar{v}, u_1, u_2, u_3, ..., v_1, v_2, v_3, ..., \bar{u}_1, \bar{u}_2, \bar{u}_3, ..., \bar{v}_1, \bar{v}_2, \bar{v}_3, ...) = 0 \\ E_2(x, t, u, v, \bar{u}, \bar{v}, u_1, u_2, u_3, ..., v_1, v_2, v_3, ..., \bar{u}_1, \bar{u}_2, \bar{u}_3, ..., \bar{v}_1, \bar{v}_2, \bar{v}_3, ...) = 0\,. \end{aligned} \tag{7}$$

The above form explains why we have restricted the generators η and ξ not to depend on derivatives with respect to t.

The identities E_1 and E_2 are polynomials in the indicated variables which we regard as independent. That is, identities (7) must hold for all values of x, t, u, v, $\bar{u}$, $\bar{v}$, u_1, u_2, $u_3, \ldots$. Setting, successively, the coefficients of these variables (including powers and products between them) equal to zero we obtain a large number of partial differential equations in η and ξ which need to be satisfied. Therefore these equations enable us to derive the generators η and ξ and consequently the desired Lie–Bäcklund transformations.

We split the following analysis into two exclusive cases: (i) $h \neq 1$ and (ii) $h = 1$.

Case (i). Without presenting any calculations, we state that both of the generators η and ξ in transformations (2) are independent of terms u_i, $\bar{u}_i$, v_i, $\bar{v}_i$ where $i \geqslant 3$. Hence, in this case the symmetries admitted by the CNLS (1) are not proper Lie–Bäcklund symmetries which means that they are equivalent to Lie point symmetries. Using the decomposition (4) we may write the transformations in the form (3).

For equations CNLS(A) and CNLS(C) (that is, $\beta \neq 0$) we get

$$X = 2c_3t + c_4 \qquad T = c_2 \qquad U = \mathrm{i}u(c_3x + c_5) \qquad V = \mathrm{i}v(c_3x + c_5)\,. \tag{8}$$

For equations CNLS(B) the generators are given by

$$\begin{aligned} X &= c_1x + 2c_3t + c_4 \qquad T = 2c_1t + c_2 \\ U &= (\mathrm{i}c_3x + \mathrm{i}c_5 - c_1 + 2\mathrm{i}\alpha c_1t + \mathrm{i}\alpha c_2)u \\ V &= (\mathrm{i}c_3x + \mathrm{i}c_6 - c_1 - 2\mathrm{i}\alpha c_1t - \mathrm{i}\alpha c_2)v \end{aligned} \tag{9}$$

where c_i in (8) and (9) are constants. Setting $\alpha = 0$ in (9) we obtain the corresponding results for equations CNLS(D) (see [6]).

Case (ii). For this case we state that only one proper Lie–Bäcklund symmetry exists which is

$$\begin{aligned} &(6u\bar{u}u_x + 3u\bar{v}v_x + 3v\bar{v}u_x + 2u_{xxx})\frac{\partial}{\partial u} + (6u\bar{u}\bar{u}_x + 3\bar{u}v\bar{v}_x + 3v\bar{v}\bar{u}_x + 2\bar{u}_{xxx})\frac{\partial}{\partial \bar{u}} \\ &\quad +(6v\bar{v}v_x + 3v\bar{u}u_x + 3u\bar{u}v_x + 2v_{xxx})\frac{\partial}{\partial v} \\ &\quad +(6v\bar{v}\bar{v}_x + 3\bar{v}u\bar{u}_x + 3u\bar{u}\bar{v}_x + 2\bar{v}_{xxx})\frac{\partial}{\partial \bar{v}}\,. \end{aligned} \tag{10}$$

All four parts of the equation admit the above symmetry. The remaining symmetries are equivalent to Lie point symmetries. We can therefore give them in the form (3). For equations CNLS(A) the generators of the Lie point transformations are given by

$$\begin{aligned} X &= c_1x + 2c_3t + c_4 \qquad T = 2c_1t + c_2 \\ U &= \left[\mathrm{i}c_3x + \mathrm{i}c_5\frac{\beta^2}{\gamma^2} - c_1 + 2\mathrm{i}\alpha c_1t + \frac{\mathrm{i}\alpha\beta c_6}{\gamma^2} + \mathrm{i}c_8\sin 2\gamma t + \mathrm{i}c_7\cos 2\gamma t\right]\mathrm{u} \\ &\quad + \left[\mathrm{i}c_6\frac{\beta^2}{\gamma^2} + 2\mathrm{i}\beta c_1t - \frac{\mathrm{i}\alpha\beta c_6}{\gamma^2}\right. \\ &\quad \left. + \frac{\gamma}{\beta}c_7\sin 2\gamma t - \frac{\gamma}{\beta}c_8\cos 2\gamma t - \mathrm{i}\frac{\alpha}{\beta}c_8\sin 2\gamma t - \mathrm{i}\frac{\alpha}{\beta}c_7\cos 2\gamma t\right]v \end{aligned} \tag{11}$$

$$\begin{aligned} V &= \left[\mathrm{i}c_3x + \mathrm{i}c_5\frac{\beta^2}{\gamma^2} - c_1 - 2\mathrm{i}\alpha c_1t - \frac{\mathrm{i}\alpha\beta c_6}{\gamma^2} - \mathrm{i}c_8\sin 2\gamma t - \mathrm{i}c_7\cos 2\gamma t\right]v \\ &\quad + \left[\mathrm{i}c_6\frac{\beta^2}{\gamma^2} + 2\mathrm{i}\beta c_1t - \frac{\mathrm{i}\alpha\beta c_6}{\gamma^2}\right. \\ &\quad \left. + \frac{\gamma}{\beta}c_8\cos 2\gamma t - \frac{\gamma}{\beta}c_7\sin 2\gamma t - \mathrm{i}\frac{\alpha}{\beta}c_8\sin 2\gamma t - \mathrm{i}\frac{\alpha}{\beta}c_7\cos 2\gamma t\right]u \end{aligned}$$

where $\gamma = \sqrt{\alpha^2+\beta^2}$. In order to obtain the desired results for equations CNLS(C) we simply set $\alpha = 0$ in (11). To derive the transformations (11) we have assumed that $\beta \neq 0$. Therefore, we cannot obtain the corresponding results for equations CNLS(B) and CNLS(D) from (11).

For equations CNLS(B) we get the transformations

$$\begin{aligned} X &= c_1 x + 2c_3 t + c_4 \qquad T = 2c_1 t + c_2 \\ U &= (\mathrm{i}c_3 x + \mathrm{i}c_5 - c_1 + 2\mathrm{i}\alpha c_1 t)u + (c_7 + \mathrm{i}c_8)\mathrm{e}^{2\mathrm{i}\alpha t} v \\ V &= (\mathrm{i}c_3 x + \mathrm{i}c_6 - c_1 - 2\mathrm{i}\alpha c_1 t)v + (-c_7 + \mathrm{i}c_8)\mathrm{e}^{-2\mathrm{i}\alpha t} u\,. \end{aligned} \tag{12}$$

Setting $\alpha = 0$ in (12) we obtain the corresponding generators for equations CNLS(D) (see [6]).

3. Similarity solutions

We now solve the characteristics equations

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}u}{U} = \frac{\mathrm{d}v}{V} \tag{13}$$

which enable us to obtain the similarity transformations which reduce the CNLS into a system of ordinary differential equations. We shall present only the results for CNLS(B) in the case where $h \neq 1$. In a similar manner the corresponding similarity solutions for the remaining CNLS equations may be obtained. The analysis is more complicated in the cases where the generators are given by (11) and (12).

The first equality in (13) give a solution of the form $\eta(x,t) =$ constant, where η will be the independent variable of the resulting ordinary differential equations. Although the general solution to this linear equation may be found formally, it is better to analyse the various possibilities separately. We therefore split the analysis into five exclusive cases:(i) $c_1 = c_3 = 0,\ c_2 \neq 0$, (ii) $c_1 = c_2 = 0$, (iii) $c_1 = 0,\ c_3 \neq 0$, (iv) $c_1 \neq 0,\ c_3 = 0$ and (v) $c_1 \neq 0,\ c_3 \neq 0$.

Case (i). We take $c_1 = c_2 = 0,\ c_4/c_2 = 2c,\ c_5/c_2 = \gamma_1,\ c_6/c_2 = \gamma_2$. Substitution of (9) into (13) gives

$$\frac{\mathrm{d}x}{2c} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}u}{\mathrm{i}(\gamma_1+\alpha)u} = \frac{\mathrm{d}v}{\mathrm{i}(\gamma_2-\alpha)v}\,.$$

Solving the above equations, we obtain $\eta = x - 2ct$ and

$$\begin{aligned} u &= \mathrm{i}F(\eta)\exp\{\mathrm{i}[(\gamma_1+\alpha)t + c\eta + \delta_1]\} \\ v &= \mathrm{i}G(\eta)\exp\{\mathrm{i}[(\gamma_2-\alpha)t + c\eta + \delta_2]\} \end{aligned}$$

where we assume that the functions F and G are real and δ_1 and δ_2 are real constants. Upon substituting this similarity solution into CNLS(B) we get

$$\begin{aligned} &F'' - (F^2 + hG^2 + c^2 - k_1)F = 0 \\ &G'' - (G^2 + hF^2 + c^2 - k_2)G = 0\,. \end{aligned}$$

The above system of ordinary differential equations also arises for CNLS(D) [6] and has been studied numerically by Parker and Sophocleous [7, 8].

Case (ii). Since the parameter α disappears from (9), the similarity solution is expected to be the same as for CNLS(D). If $c_3 \neq 0$, there is no loss of generality in shifting the origin to take $c_4 = 0$. The corresponding equations (13) give the solution $\eta = t$ and

$$u = F(\eta)\exp\{\mathrm{i}[\tfrac{1}{4}t^{-1}(x+\gamma_1)^2 + \delta_1 \ln t + \delta_2]\}$$
$$v = G(\eta)\exp\{\mathrm{i}[\tfrac{1}{4}t^{-1}(x+\gamma_2)^2 + \delta_3 \ln t + \delta_4]\}$$

where $\gamma_1 = c_5/c_3$, $\gamma_2 = c_6/c_3$ which reduces the CNLS(B) into

$$\mathrm{i}F' + \tfrac{1}{2}\mathrm{i}\eta^{-1}F + (-\delta_1\eta^{-1} + F\bar{F} + hG\bar{G} + \alpha)F = 0$$
$$\mathrm{i}G' + \tfrac{1}{2}\mathrm{i}\eta^{-1}G + (-\delta_3\eta^{-1} + G\bar{G} + hF\bar{F} - \alpha)G = 0\,.$$

The assumption that the functions $F(\eta)$ and $G(\eta)$ are real gives easily the general solution for the above system and consequently we have a solution for the CNLS(B).

Now, if we take $c_3 = 0$ we obtain the similarity solution $\eta = t$ and

$$u = F(\eta)\exp[\mathrm{i}(\gamma_1 x + \delta_1 t + \delta_2)] \qquad v = G(\eta)\exp[\mathrm{i}(\gamma_2 x + \delta_3 t + \delta_4)]$$

where $\gamma_1 = c_5/c_4$, $\gamma_2 = c_6/c_4$ which transforms the CNLS(B) into

$$\mathrm{i}F' + (F\bar{F} + hG\bar{G} + \alpha - \delta_1 - \gamma_1^2)F = 0$$
$$\mathrm{i}G' + (G\bar{G} + hF\bar{F} - \alpha - \delta_3 - \gamma_2^2)G = 0\,.$$

Case (iii). Without loss of generality we take $c_4 = 0$ and set $c_3/c_2 = c$, $c_5/c_2 = \gamma_1$ and $c_6/c_2 = \gamma_2$. In this case we obtain the similarity solution $\eta = x - ct^2$ and

$$u = F(\eta)\exp[\mathrm{i}(cxt - \tfrac{2}{3}c^2t^3 + \gamma_1 t + \alpha t)]$$
$$v = G(\eta)\exp[\mathrm{i}(cxt - \tfrac{2}{3}c^2t^3 + \gamma_2 t - \alpha t)]$$

which reduces the CNLS(B) into the system

$$F'' + (F\bar{F} + hG\bar{G} - c\eta - \gamma_1)F = 0$$
$$G'' + (G\bar{G} + hF\bar{F} - c\eta - \gamma_2)G = 0\,.$$

Case (iv). Without loss of generality we take $c_2 = c_4 = 0$ and set $c_5/c_1 = \gamma_1$ and $c_6/c_1 = \gamma_2$. The desired similarity solution is given by $\eta = x/\sqrt{t}$ and

$$u = \frac{1}{\sqrt{t}}F(\eta)\exp[\mathrm{i}(\tfrac{1}{2}\gamma_1 \ln t + \alpha t + \tfrac{1}{8}\eta^2 + \delta_1)]$$
$$v = \frac{1}{\sqrt{t}}G(\eta)\exp[\mathrm{i}(\tfrac{1}{2}\gamma_2 \ln t - \alpha t + \tfrac{1}{8}\eta^2 + \delta_2)]$$

which transforms the CNLS(B) into

$$F'' + (F\bar{F} + hG\bar{G} + \tfrac{1}{16}\eta^2 - \tfrac{1}{2}\gamma_1 - \tfrac{1}{4}\mathrm{i})F = 0$$
$$G'' + (G\bar{G} + hF\bar{F} + \tfrac{1}{16}\eta^2 - \tfrac{1}{2}\gamma_2 - \tfrac{1}{4}\mathrm{i})G = 0\,.$$

Case (v). Similarly as in case (iv) we take $c_2 = c_4 = 0$, $c_3/c_1 = c/2$, $c_5/c_1 = \gamma_1$ and $c_6/c_1 = \gamma_2$. The similarity solution is given by $\eta = x - ct/\sqrt{t}$ and

$$u = \frac{1}{\sqrt{t}}F(\eta)\exp[\mathrm{i}(\tfrac{1}{2}cx - \tfrac{1}{4}c^2t + \tfrac{1}{2}\gamma_1 \ln t + \alpha t + \tfrac{1}{8}\eta^2 + \delta_1)]$$
$$v = \frac{1}{\sqrt{t}}G(\eta)\exp[\mathrm{i}(\tfrac{1}{2}cx - \tfrac{1}{4}c^2t + \tfrac{1}{2}\gamma_2 \ln t - \alpha t + \tfrac{1}{8}\eta^2 + \delta_2)]$$

which reduces the CNLS(B) into the same system of ordinary differential equations which was obtained in case (iv).

4. Remarks

The procedure which we have used to obtain the infinitesimal transformations (8) and (9) is known as *Lie classical method* [9]. Bluman and Cole [10] proposed a generalization of the Lie method which is known as *non-classical method.* A further generalization of the latter method is presented by Olver and Rosenau [11, 12]. Recently, Clarkson and Kruskal [13] introduced a direct method for finding similarity solutions without using transformation group theory. Originally this method seemed to produce results that could not be obtained by any other method, but later it appeared that this is not the case [14–16]. The same results can be obtained by the non-classical method. In fact, the non-classical method of Bluman and Cole is more general than the direct method of Clarkson and Kruskal.

The objective of these new methods is to derive symmetries which cannot be obtained by the classical method. We state that for the CNLS (1) the non-classical method only produces the infinitesimal transformations (8) and (9). We also state that if α and β in CNLS (1) are functions of x and t, symmetry (10) is the only proper Lie–Bäcklund symmetry admitted by CNLS (1).

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